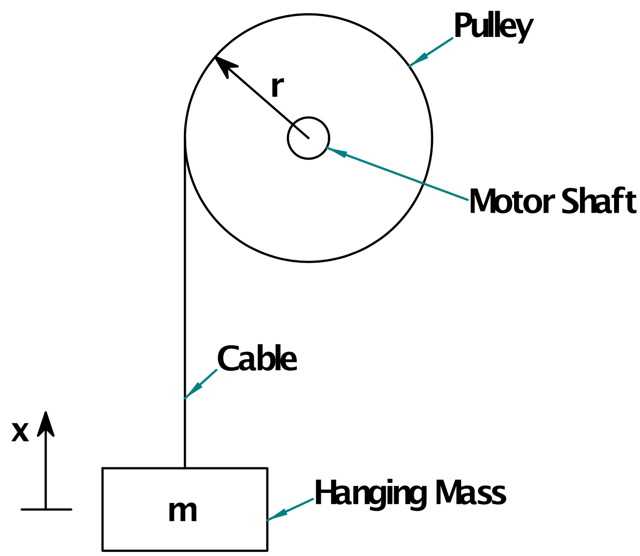
Dynamic Modeling and Optimization of a Mass Raised by a Pulley

Overview

This example analyzes and optimizes the system shown below. A pulley is attached to a motor shaft, and a cable from the pulley supports a hanging mass. As the pulley is rotated the mass is raised. The design objective is to raise the mass a specified distance in as short a time as possible. The motor is a DC brush motor with a linear torque-speed curve. The design parameter that is to be optimized is the pulley radius; a small radius will provide a larger initial acceleration, yet a larger radius can provide a higher final velocity. The analysis presented here will identify the optimal radius.

System Parameters: m => mass x => distance mass is raised I => combined moment of inertia of pulley, motor shaft, and motor rotor. r => radius of pulley θ => angle of rotation of pulley τ_m => motor torque $\tau_{m,stall}$ => stall torque of motor ω_{no_load} => no load speed	

Assumptions

- It is assumed that the motor is operated at its maximum voltage throughout the lifting process. This approach will indeed identify the fastest lifting time, but if precise positioning of the mass is required than a variable voltage will be required for control purposes such as with PID control. The maximum voltage assumption is a good method for identifying the optimal pulley size, but in a precision lifting application the actual lifting speed will be lower due to control factors. In addition, if the mass is to be positioned precisely, then a deceleration period should also be incorporated into the analysis.

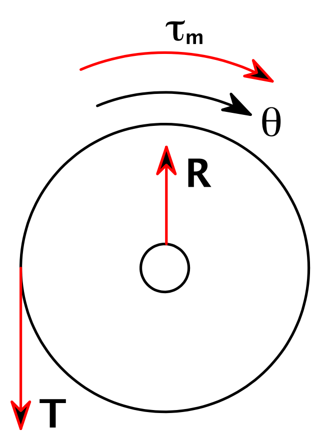
- In this analysis it is assumed that there are no frictional losses. However, when the actual frictional loses are determined, they can be incorporated by subtracting them from the motor torque.

Optimization Approach

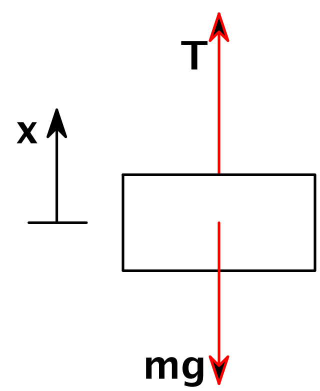
The optimization approach has the following steps:

1. Model system dynamics for a given pulley radius.
2. Validate the model by comparing predictions to experimental results. This step is not part of this write-up, but is an essential step in optimization.
3. Run the simulation for various pulley radiuses to identify the optimal radius.

Free Body Diagrams (FBDs)



FBD of Pulley



FBD of Mass

Equations of Motion

The pulley is a rigid body rotating about a pivot, and thus its equation of motion is given by the sum of the moments about the pivot as:

$$\tau_m - Tr = I\ddot{\theta} \qquad \text{(based upon FBD of pulley)} \qquad \text{eq. 1}$$

where T is the cable tension. Note, it is a mistake to assume that T=mg, since this is only valid under static conditions.

The equation of motion of the mass is:

$$T - mg = m\ddot{x} \qquad \text{(based upon FBD of mass)} \qquad \text{eq. 2}$$

The mass is raised as the cable is wrapped around the pulley. Thus, the geometric relationship between the $\dot{x}$ and $\dot{\theta}$, is given below where $\dot{\theta}$ is in radians/second.

$$\dot{x} = r\dot{\theta} \qquad \text{eq. 3}$$

Taking the time derivative of both sides of equation 3 provides:

$$\ddot{x} = r\ddot{\theta} \qquad \text{eq. 4}$$

The cable tension, T, is the common element between equation 1 and 2, and indeed is the element that connects the pulley and mass physically. Thus, solving for T allows us to combine equations 1 and 2. From equation 1 we get:

$$T = \frac{1}{r}\left(\tau_m - I\ddot{\theta}\right) \qquad \text{eq. 5}$$

From equation 2, we get:

$$T = m\ddot{x} + mg \qquad \text{eq. 6}$$

Setting the right side of equations 5 and 6 equal to each other provides:

$$\frac{1}{r}\left(\tau_m - I\ddot{\theta}\right) = m\ddot{x} + mg \qquad \text{eq. 7}$$

Substituting equation 4 into equation 7 and solving for $\ddot{\theta}$ gives:

$$\ddot{\theta} = \frac{\tau_m - mgr}{I + mr^2} \qquad \text{eq. 8}$$

In the above equation the term, $I + mr^2$, can be interpreted as an equivalent inertia, which is accelerated by the overall torque in the system. (Note, in a system with multiple rotating gears or pulleys, an equivalent inertia term also appears) All that remains to calculate $\ddot{\theta}$ at any given time is to determine the motor torque.

Motor Torque

For a DC brush motor the motor torque-speed curve is a straight line, as described by in the Primer on DC Motors by David Gowden. When the voltages applied to the motor vary, it is necessary to identify all the motor constants. However, in this case we assume that the motor is operated at a fixed maximum voltage; the maximum torque is $\tau_{m,stall}$ when the angular velocity is zero, and the maximum angular velocity is ω_{no_load} which occurs when the torque is zero. This relationship can be written as:

$$\tau_m = \tau_{m,\,stall} - \omega \frac{\tau_{m,\,stall}}{\omega_{no\_load}} \qquad \text{eq. 9}$$

where, ω is the angular velocity in radians per second.

Solving the Differential Equation with the Euler Method

The equation of motion (eq. 8) is a differential equation which must be solved in time to predict the performance of the system. One method is the Euler method. This method is not the most efficient or accurate, but is easy to implement numerically. Starting with the initial state variables, $\theta(t)$ and $\dot{\theta}(t)$, a small step in time δt is taken. The state variables at the next time step are given by:

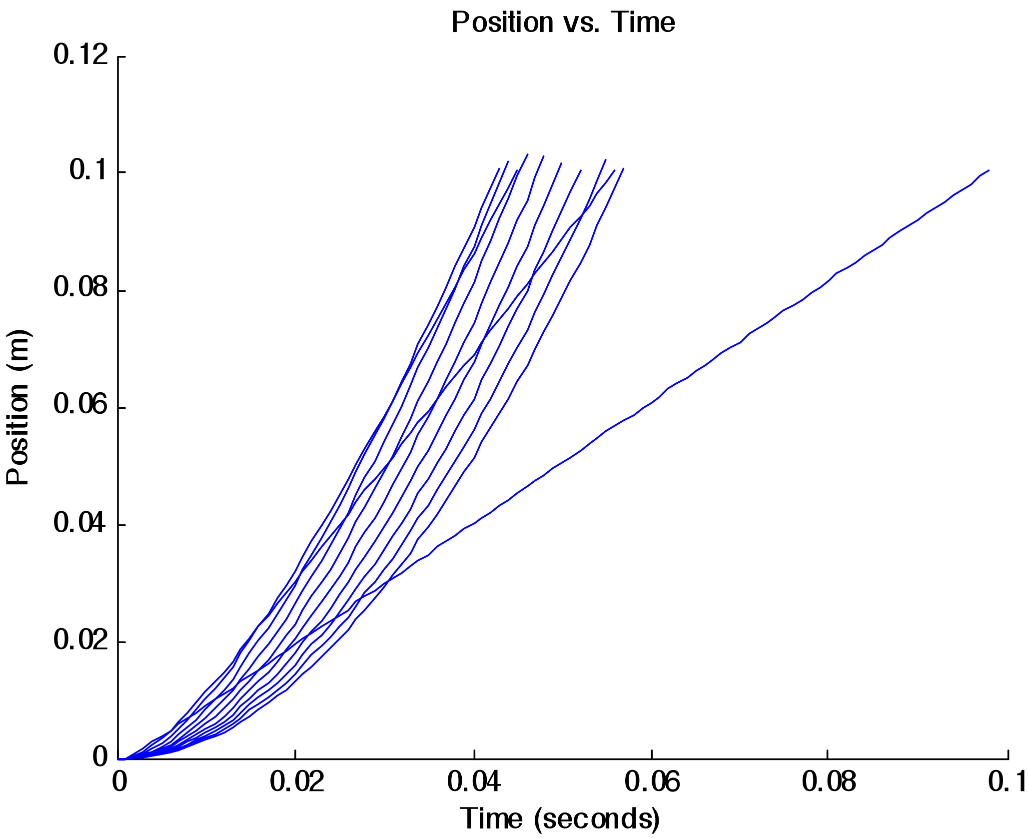
$$\theta(t + \delta t) \approx \theta(t) + \dot{\theta}(t)\delta t \qquad \text{eq. 10}$$

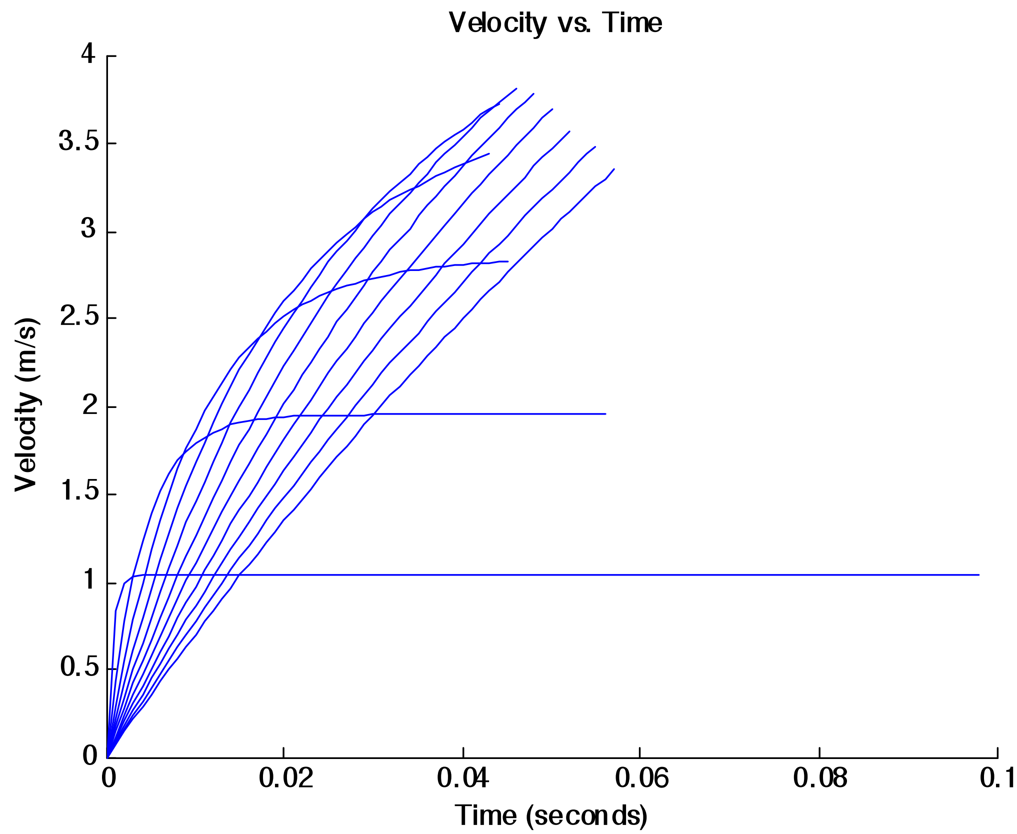
$$\dot{\theta}(t + \delta t) \approx \dot{\theta}(t) + \ddot{\theta}(t)\delta t$$

At each time step the motor torque is calculated using eq. 9, where $\omega = \dot{\theta}$. The equation of motion (eq. 8) can then be solved to get $\ddot{\theta}$. Finally the state variables at the next time step are calculated using eq. 10.

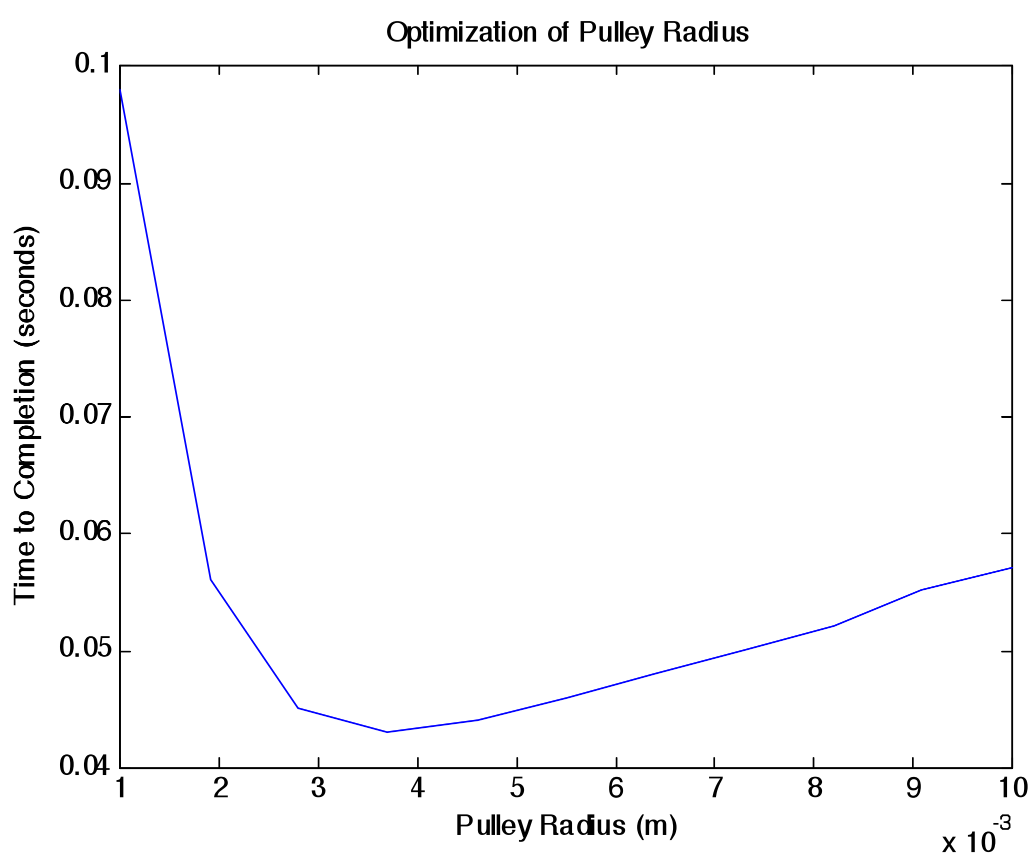
Modeling and Optimization of a Pulley Raising a Mass

Results of Matlab simulation





Velocity vs. Time
Velocity (m/s)
Time (seconds)
0
0.5
1
1.5
2
2.5
3
3.5
4
0.02
0.04
0.06
0.08
0.1



Optimization of Pulley Radius
Time to Completion (seconds)
Pulley Radius (m)
x 10^-3
0.04
0.05
0.06
0.07
0.08
0.09
0.1
1
2
3
4
5
6
7
8
9
10